SUM RULES FOR THE DIRAC SPECTRUM OF THE SCHWINGER MODEL

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The inverse eigenvalues of the Dirac operator in the Schwinger model satisfy the same Leutwyler-Smilga sum rules as in the case of QCD with one flavor. In this paper we give a microscopic derivation of these sum rules in the sector of arbitrary topological charge. We show that the sum rules can be obtained from the clustering property of the scalar correlation functions. This argument also holds for other theories with a mass gap and broken chiral symmetry such as QCD with one flavor. For QCD with several flavors a modified clustering property is derived from the low energy chiral Lagrangian. We also derive sum rules for a fixed external gauge field. Both types of sum rules are obtained from the bosonized version of the Schwinger model.

In the sector of topological charge ν the sum rules are consistent with a shift of the Dirac spectrum away from zero by $\nu/2$ average level spacings. This shift is also required to obtain a nonzero chiral condensate in the massless limit. Finally, we discuss the Dirac spectrum for a closely related two-dimensional theory with a gauge field action that is quadratic in the gauge field and has applications to the quantum Hall effect and d-wave super-conductors.

Contents

I. INTRODUCTION

The Schwinger model [1], electrodynamics in $1+1$ dimension is one of the few quantum field theories that can be solved analytically [2, 3, 4]. Since the model contains nonperturbative features that are also found in QCD, such as a nonzero chiral condensate and instantons, it has added significantly to our understanding of gauge theories in general. However, most interesting quantum field theories cannot be solved analytically, and one has to rely on approximate techniques to investigate them. For nonperturbative phenomena, two approaches which both have been tested in the Schwinger model, have been very successful: lattice simulations and chiral perturbation theory. In particular, a large number of lattice studies have benefited from the Schwinger model [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Among others, we mention the study of chiral fermions on the lattice [8, 9, 10, 14], the analysis of the continuum limit for staggered fermions [17], and the effect of quenching on the Dirac spectrum [18, 19].

For theories with a mass gap and a nonzero chiral condensate the mass dependence of the partition function can be obtained from a chiral Lagrangian. In particular, in the domain where the Compton wavelength of the Goldstone modes is much larger than the size of the box only the constant fields have to be included in the partition function so that its mass dependence simply follows from the mass term of the chiral Lagrangian [20]. Based on this result, it was shown in [20] that the Dirac spectrum has to satisfy consistency relations in the form of sum rules for inverse eigenvalues. These relations are not sufficient to determine the Dirac spectrum. However, it turned out that the complete low-lying Dirac spectrum can be obtained from a modified chiral Lagrangian [21, 22, 23], or equivalently, from chiral random matrix theory [24, 25]. In the Schwinger model, and also for QCD with $N_f = 1$, there are no low-lying excitations and the only contribution to the low-energy limit of the partition function is the vacuum energy. Starting from this result, it is possible to derive sum rules for the inverse Dirac eigenvalues of the Schwinger model [20, 26]. However, to obtain a deeper understanding of these sum rules in the context of quantum field theory, it of is interest to derive them directly from the microscopic field theory. This was achieved in [26] for the simplest sum rule in the sector of zero topological charge.

In this paper we give a microscopic derivation of the simplest sum rule in the sector of

arbitrary topological charge ν. This requires a detailed understanding of scalar correlation functions in the Schwinger model which were derived before both in the plane [27] and on the torus [4, 28, 29, 30, 31]. It turns out that the clustering property of the scalar correlation functions alone is sufficient to derive the (gauge field averaged) sum rules. Below we will give a derivation based on this property. A third way to obtain Leutwyler-Smilga sum rules is to start from the bosonized form of the massive Schwinger model. In the microscopic derivation of sum rules we first derive a more informative quantity – the same sum rules for an arbitrary fixed external gauge field (which was also obtained for $\nu = 0$ in [26]). The universal Leutwyler-Smilga sum rules then follow after averaging over the gauge fields.

Analytical results obtained for the Schwinger model have been tested elaborately by means of Monte-Carlo simulations. The numerical value of the chiral condensate has been reproduced accurately [7, 17, 18]. The index theorem was confirmed and the fermionic would-be zero modes were identified [9, 10]. Leutwyler-Smilga sum rules have been obtained both for zero and nonzero topological charge [10]. The distributions of the smallest eigenvalues of the Dirac operator for the Schwinger model [7, 10, 17, 18] agree with analytical results obtained by means of chiral Lagrangians [20, 26] and random matrix theory [24, 25, 32].

A theory that is closely related to the Schwinger model are random Dirac fermions where the electromagnetic interaction is replaced by a random gauge field interaction. Starting with the work of Gade and Wegner [33] the behavior of the spectral density near zero has attracted a great deal of attention in particular with applications to the quantum Hall effect in mind [34, 35]. An important difference with the Schwinger model is that in the condensed matter literature the physically interesting case is the quenched model, sometimes defined as the limit of zero flavors. The consensus is that the density of states diverges for strong disorder as is the case for the quenched Schwinger model [18, 26, 36, 37]. For weak disorder, on the other hand, the spectral density vanishes. We will discuss the Dirac spectrum of this model and relate it to the chiral condensate and the simplest Leutwyler-Smilga sum rule.

The organization of this paper is as follows. In an introductory chapter we introduce the Leutwyler-Smilga sum rules and the Schwinger model and discuss its main properties. A

combinatorial derivation of sum rules based on the clustering property of scalar correlators is given in section III. Section IV contains a microscopic derivation of the simplest sum rule, and a derivation from the bosonized massive Schwinger model is given in section V. It is explained in section VI that modified sum rules for two or more flavors can be related to modified clustering properties of the scalar correlators. A random matrix derivation of sum rules is given in section VII. In this section we also study the fluctuations of the fermion determinant. In the first part of section VIII we discuss the effect of topology on the Dirac spectrum and the mechanism of the formation of a chiral condensate. In the second part of section VIII we discuss the behavior of the Dirac spectrum for random fermions. Concluding remarks are made in section IX.

II. GENERAL FRAMEWORK

A. Sum Rules

In this paper we study the eigenvalues of the Dirac operator by means of sum rules for its inverse eigenvalues introduced in [20]. The eigenvalues of the anti-hermitian Dirac operator are defined by

$$D\phi_k = i\lambda_k \phi_k. \tag{1}$$

In sector with topological charge ν the Dirac operator has exactly ν zero eigenvalues that are not paired. All other eigenvalues occurs in pairs $\pm\lambda_k$. In the sector of topological charge ν the partition function is defined by

$$Z^\nu(m) = m^\nu \int dA_\mu^\nu \det{}'[iD] e^{-S(A_\mu^\nu)}, \tag{2}$$

where the integral is over gauge fields in the sector of topological charge ν weighted by the gauge field action $S(A_\mu)$. As indicated by the prime,

$$\det{}'[iD] = \prod_{\lambda_n \neq 0} \lambda_n^2, \tag{3}$$

the zero eigenvalues are not included in the determinant resulting in the pre-factor m^ν. At fixed θ angle the partition function is given by

$$Z(m,\theta) = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} Z^\nu(m). \tag{4}$$

Below we will use two types of averages of an operator O denoted by single and double brackets. The first type is defined by (notice that $Z^0(m=0) = Z(m=0,\theta)$)

$$\langle O\rangle^\nu = \frac{1}{Z^0(m=0)} \int DA_\mu^\nu O[A_\mu^\nu] {\rm det}'(iD) e^{-S(A_\mu^\nu)}, \tag{5}$$

and the second type by [20]

$$\langle\langle O\rangle\rangle^\nu = \left.\frac{m^\nu}{Z^\nu(m)}\right|_{m=0} \int DA_\mu^\nu O(A_\mu^\nu) {\rm det}'(iD) e^{-S(A_\mu^\nu)}. \tag{6}$$

To obtain sum rules for the inverse Dirac eigenvalues in a sector of topological charge ν we expand the fermion determinant in powers of m,

$$ {\rm det}'[i\not{D} + m] = {\rm det}'[i\not{D}] \prod_{n\neq 0} \Big(1 + m^2 \sum_{n>0} \frac{1}{\lambda_n^2} + m^4 \sum_{\substack{n_i>0 \\ n_1\neq n_2}} \frac{1}{\lambda_{n_1}^2\lambda_{n_2}^2} + \cdots\Big). \tag{7}$$

For the small mass expansion of the partition function we thus obtain

$$\begin{aligned} \frac{Z^\nu(m)}{Z^0(m=0)} &= m^\nu \left\langle {\rm det}'[iD]\right\rangle^\nu + m^{\nu+2} \left\langle \sum_{n>0} \frac{1}{\lambda_n^2} \right\rangle^\nu + m^{\nu+4} \left\langle \sum_{\substack{n_i>0 \\ n_1\neq n_2}} \frac{1}{\lambda_{n_1}^2\lambda_{n_2}^2} \right\rangle^\nu + \cdots \\ &= m^\nu \left\langle {\rm det}'[iD]\right\rangle^\nu \Big[1 + m^2 \left\langle\left\langle \sum_{n>0} \frac{1}{\lambda_n^2} \right\rangle\right\rangle^\nu + m^4 \left\langle\left\langle \sum_{\substack{n_i>0 \\ n_1\neq n_2}} \frac{1}{\lambda_{n_1}^2\lambda_{n_2}^2} \right\rangle\right\rangle^\nu + \cdots\Big]. \end{aligned} \tag{8}$$

where we have used the definition (6) for the average in the sector of topological charge ν. Therefore, sum rules for the inverse Dirac eigenvalues follow from the sub-leading terms in the small mass expansion of the partition function in the sector of topological charge ν. If

$$\frac{Z^\nu(m)}{Z^0(m=0)} = m^\nu (a_0 + a_2 m^2 + \cdots), \tag{9}$$

the simplest sum rule is given by

$$\left\langle\left\langle \sum_{n>0} \frac{1}{\lambda_n^2} \right\rangle\right\rangle^\nu = \frac{a_2}{a_0}. \tag{10}$$

In [20] general arguments where given that, for a gauge theory interacting with fermions according to the Dirac operator with a nonzero chiral condensate, this ratio is given by

$$\frac{\Sigma^2 V^2}{4(N_f+\nu)}. \tag{11}$$

For $N_f = 1$ this argument is particularly simple. The θ dependence of the partition function, as can be observed from (2) and (4), is obtained from the replacement $m \to$

$m\exp(i\theta)$. Given that the vacuum energy is equal to $-\mathrm{Re}(mV\Sigma)$, this results in the large volume partition function [20]

$$Z(m,\theta) = e^{mV\Sigma\cos\theta}. \tag{12}$$

By inverting (4) we find that the partition function in sector ν is given by [20]

$$Z_\nu = I_\nu(mV\Sigma) = \frac{1}{\nu! 2^\nu}(mV\Sigma)^\nu (1 + \frac{(mV\Sigma)^2}{4(|\nu|+1)} + \cdots). \tag{13}$$

This results immediately in the sum rule (11). In this paper we will derive this sum rule by a microscopic calculation in the Schwinger model.

B. The Schwinger Model

In this section we give a brief review of the Schwinger Model [1] which is massless QED in two dimensions. The Euclidean Lagrangian is defined by

$$\mathcal{L} = \frac{1}{2}F_{01}^2 - \bar{\psi}[i\,\partial\!\!\!/ - g\,A\!\!\!/]\psi, \tag{14}$$

where the electric field strength is given by

$$F_{01} = \partial_0 A_1 - \partial_1 A_0, \tag{15}$$

and

$$i\,\partial\!\!\!/ - g\,A\!\!\!/ = \gamma_\mu(i\partial_\mu - gA_\mu), \tag{16}$$

with Euclidean gamma matrices defined by

$$\gamma_0 = \sigma_1, \qquad \gamma_1 = \sigma_2, \qquad \gamma_5 = \sigma_3. \tag{17}$$

The Lagrangian of this model has a chiral symmetry which is broken by the $U(1)$ axial anomaly. As a consequence the "photon" becomes massive with mass given by $\mu = g/\sqrt{\pi}$. In this model, a local external charge is screened by massless fermions. Therefore, the asymptotic states contain no fermions, and the Schwinger boson is the only physical particle. The theory is super-renormalizable with a coupling constant g (which has the dimension of mass) that does not run.

The (massless) Schwinger model, being equivalent to a noninteracting gas of bosons, is exactly solvable and was solved many times, by different techniques (operator language, path integral, bosonization) and on different manifolds [2, 3, 4, 38, 39, 40, 41, 42, 43]. Also, many authors considered more specific features of the model, such as certain correlation functions at zero and at finite temperature [27, 28, 29, 30, 31, 44, 45, 46, 47], the fermion determinant, zero modes, the index theorem and instantons [4, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57], the multi-flavor Schwinger model [58, 59, 60, 61, 62, 63].

The vector potential A_μ can be decomposed as

$$A_\mu = -\epsilon_{\mu\nu}\partial^\nu \phi + \partial_\mu \lambda, \tag{18}$$

with the last term being a pure gauge. The topological charge, ν, of the gauge fields is equal to the difference of the number of right handed and left handed zero modes of the Dirac operator and is therefore necessarily quantized. It is given by

$$\nu = \frac{g}{4\pi}\int d^2x \epsilon_{\mu\nu} F_{\mu\nu} = \frac{g}{2\pi}\int d^2x \partial_\alpha^2 \phi. \tag{19}$$

By Stokes theorem the topological charge is determined by the asymptotic behavior of ϕ. One easily shows that the large distance asymptotics,

$$\phi(x) \sim \frac{\nu}{2g}\log x^2 + \text{const.} \quad , \quad |x| \to \infty, \tag{20}$$

results in a topological charge equal to ν. The boundary conditions (20) imply that the plane is compactified at infinity, for example by stereographic projection to a sphere with radius R [49, 50, 52].

Because of a particular property of the 2d Dirac algebra,

$$\gamma_\mu \gamma_5 = -i\epsilon_{\mu\nu}\gamma_\nu, \tag{21}$$

the Dirac operator can be written in in the form [58]

$$i\, \not{D}_\phi = e^{g\phi\gamma_5} i\, \not{\partial} e^{g\phi\gamma_5}. \tag{22}$$

Using this representation one easily finds the following explicit expressions for the zero modes in the sector of topological charge ν:

$$\psi_p(x) = \frac{1}{\sqrt{2\pi}}(x^+)^p e^{-g\phi(x)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad p = 0 \cdots \nu - 1, \qquad x^\pm = x_0 \pm i x_1. \tag{23}$$

Note that these zero modes are not normalized. From the asymptotic behavior (20), it is easy to see that for $p = 0, \cdots, \nu - 2$ the zero modes are normalizable. However, the zero mode for $p = \nu - 1$ is not normalizable on the plane (its norm diverges logarithmically). It has to be included nevertheless because it is normalizable on the compactified plane as required by the index theorem [44, 49, 50, 52, 64]. The pre-factor $1/\sqrt{2\pi}$ in the zero modes also follows from matching with compact manifolds [57].

The unpaired zero modes explicitly break the $U_A(1)$ symmetry of the partition function. In the massless limit the chiral condensate $\langle\bar{\psi}\psi\rangle$ comes from the zero modes in the sectors $\nu = \pm 1$. It has a particularly simple form on the plane [2, 4, 57, 65],

$$\Sigma \equiv \langle\bar{\psi}\psi\rangle = -\frac{\mu}{2\pi} e^{\gamma}, \tag{24}$$

where γ is the Euler constant. As a consequence of the Banks-Casher formula [66] the average density of the low-lying Dirac eigenvalues is given by $\Sigma V/\pi$. This also implies that if we take the thermodynamic limit before the chiral limit, we find a nonzero value for the condensate from the nonzero modes [16].

C. Effective Action

Using the action (14) and the decomposition (18), the partition function in the sector of topological charge ν is given by

$$\begin{aligned} Z^\nu(m) &= \int D\phi^\nu e^{-S(\phi^\nu)} \int D\bar{\psi} D\psi e^{-\int d^2x \bar{\psi}[i\,\not{D}+m]\psi} \\ &= \int D\phi^\nu e^{-S(\phi^\nu)} \det[iD + m] \\ &= \int D\phi^\nu e^{-S(\phi^\nu)} m^\nu \prod_{n>0} \lambda_n^2. \end{aligned} \tag{25}$$

Here we have used the chiral symmetry of the nonzero Dirac eigenvalues. The infinite product over the eigenvalues has to be regularized in the UV. This procedure is well known [67], and results in an anomalous contribution to the effective action. The result is [4, 43, 52]

$$\prod_{n>0} \lambda_n^2 = \mathcal{C} \det \mathcal{N} \exp\Big(\frac{\mu^2}{2} \int d^2x \phi(x) \Delta\phi(x)\Big), \tag{26}$$

where $\mathcal{C}$ is an (infinite) normalization constant which drops out from all final results and $\mathcal{N}$ is the norm matrix of the (unnormalized) zero modes,

$$\mathcal{N}_{pq} = \int d^2x \psi_p^\dagger(x)\psi_q(x). \tag{27}$$

The bosonic partition function in the sector of topological charge ν is thus given by (to the lowest order in m)

$$Z^\nu(m) = m^\nu \int D\phi^\nu \det \mathcal{N} e^{-\Gamma[\phi^\nu]}, \tag{28}$$

with the effective action Γ equal to

$$\Gamma(\phi) = \frac{1}{2}\int d^2x \phi(x)[\Delta^2 - \mu^2\Delta]\phi(x). \tag{29}$$

This effective action Γ defines the propagator

$$\frac{1}{p^4 + p^2\mu^2} = \frac{1}{\mu^2}\Big(\frac{1}{p^2} - \frac{1}{p^2+\mu^2}\Big). \tag{30}$$

In the coordinate space,

$$\mathcal{G}(x) = -\frac{1}{4g^2}\log x^2 - \frac{1}{2g^2}K_0(\mu|x|) + \text{const.}, \tag{31}$$

where $K_0(\mu|x|)$ is a modified Bessel function of the second kind, which is exponentially small at large distances. The large distance behavior of this propagator is therefore determined by the first term which is a massless propagator

$$\mathcal{G}(x) = -\frac{1}{4g^2}\log x^2 + \text{const.}. \tag{32}$$

The arbitrary constant in the equation (31) can only be obtained after a suitable infrared regularization of the theory. For example, this constant can be determined by matching with the result on compact manifolds [56, 57].

D. Partition Function in the Sector of Topological Charge ν

To evaluate the partition function in the sector of topological charge ν (which we take positive for notational convenience; of course, the partition function does not depend on

the sign of the topological charge), we first rewrite the norm matrix. Using the explicit expressions for the zero modes we find

$$\det\mathcal{N} = \tfrac{1}{(2\pi)^\nu}\int d^2x_1\cdots d^2x_\nu \frac{1}{\nu!}\sum_{\sigma\pi}\mathrm{sg}(\sigma\pi)x_1^{*\,\sigma(0)}x_1^{\pi(0)}\cdots x_\nu^{*\sigma(\nu-1)}x_\nu^{\pi(\nu-1)}e^{-2g\sum_{q=1}^\nu\phi(x_q)}. \tag{33}$$

The sums over permutations σ and π can be rewritten as the product of two determinants,

$$\begin{aligned}\det\mathcal{N} &= \frac{1}{\nu!(2\pi)^\nu}\int d^2x_1\cdots d^2x_\nu e^{-2g\sum_{q=1}^\nu\phi(x_q)}\begin{vmatrix}1 & x_1 & \cdots & x_1^{\nu-1}\\ \vdots & \vdots & & \vdots\\ 1 & x_\nu & \cdots & x_{\nu-1}^{\nu}\end{vmatrix}^2 = \\ &= \frac{1}{\nu!(2\pi)^\nu}\int d^2x_1\cdots d^2x_\nu e^{-2g\sum_{q=1}^\nu\phi(x_q)}\prod_{i>j}^{\nu}|x_i-x_j|^2,\end{aligned} \tag{34}$$

where a well-known property of Vandermonde determinant was used. Using this expression in (2), we find for the partition function to the lowest order in m,

$$Z^\nu(m) = \frac{m^\nu}{\nu!(2\pi)^\nu}\int d^2x_1\cdots d^2x_\nu\prod_{i>j}^{\nu}|x_i-x_j|^2\int D\phi e^{-\Gamma(\phi)}e^{-2g\sum_{q=1}^\nu\phi(x_q)}. \tag{35}$$

Since the effective action is Gaussian the path integral is simply given by

$$\frac{1}{Z^0(m=0)}\int D\phi\; e^{-\Gamma(\phi)}e^{-2g\sum_{q=1}^\nu\phi(x_q)} = e^{2\nu g^2\mathcal{G}(0)+4g^2\sum_{i>j}^\nu\mathcal{G}(x_i-x_j)}. \tag{36}$$

Using the explicit expression for $\mathcal{G}$ on the plane (31) we observe that the factor $\prod_{i>j}^\nu|x_i-x_j|^2$ is canceled by the asymptotic behavior of the Greens function. Thus, in the limit of large volume

$$\frac{Z^\nu(m)}{Z^0(m=0)} = \frac{m^\nu}{\nu!}\Big(\frac{e^{2g^2\mathcal{G}(0)}V}{2\pi}\Big)^\nu + O(m^{\nu+1}). \tag{37}$$

The chiral condensate can be expressed as

$$\langle\bar\psi\psi\rangle = \frac{1}{Z^0(m=0)}\frac{1}{V}\lim_{m\to0}\frac{Z^1(m)+Z^{-1}(m)}{m}. \tag{38}$$

Since $Z^{-1}(m) = Z^1(m)$, the chiral condensate is given by

$$\Sigma = \frac{1}{\pi}e^{2g^2\mathcal{G}(0)}, \tag{39}$$

which can be used to eliminate $\mathcal{G}(0)$ from the partition function. Naively, eliminating $\mathcal{G}(0)$ in favor of the condensate is equivalent to fixing the constant in (31). Notice that only the product of the square of the normalization constant of the zero modes and $\exp(2g^2\mathcal{G}(0))$ is fixed by this condition.

III. COMBINATORIAL DERIVATION OF SUM RULES

In the sector of topological charge ν we find for the small mass expansion of the partition function in a finite but large volume V

$$\frac{Z^\nu(m)}{Z^0(m=0)} = \sum_{n=0}^{\infty} \frac{m^n}{n!} \int d^dx_1 \cdots d^dx_n \langle \bar\psi\psi(x_1) \cdots \bar\psi\psi(x_n)\rangle^\nu. \tag{40}$$

The average is over field configurations with topological charge ν weighted by the gauge field action and the determinant of the Dirac operator in the space of nonzero modes. For one flavor, the terms in the expansion with $n < \nu$ vanish, since the number of zero modes in the correlator is not sufficient to compensate the zeros of fermion determinant. For $n = \nu$ we have

$$\langle \bar\psi\psi(x_1) \cdots \bar\psi\psi(x_n)\rangle^\nu = \langle \bar\psi_R\psi_L(x_1) \cdots \bar\psi_R\psi_L(x_n)\rangle^\nu. \tag{41}$$

For massless quarks, this expectation value is called a minimal correlator and saturated by zero modes [30, 44, 68]. If we use the explicit expressions for the zero modes and take into account their (anti-)symmetrization, we recover (35) and what followed. However, since we are only interested in the leading large-volume behavior, there is a simpler way - to use the clustering property: for all $|x_i - x_j| \to \infty$,

$$\langle \bar\psi_R\psi_L(x_1) \cdots \bar\psi_R\psi_L(x_n)\rangle^\nu \to \Big(\frac{\Sigma}{2}\Big)^\nu, \tag{42}$$

so that, in agreement with (37) [69]

$$\frac{Z^\nu(m)}{Z^0|_{m=0}} = \frac{m^\nu}{\nu!}\Big(\frac{\Sigma V}{2}\Big)^\nu + O(m^{\nu+1}). \tag{43}$$

Let us consider sub-leading orders in m in (40). For $n > \nu$, the non-vanishing contributions have $n - \nu$ contractions from the nonzero mode part of the propagator. Since the massless propagator connects only states with the same chirality the expectation value $\langle \bar\psi\psi(x_1) \cdots \bar\psi\psi(x_{\nu+1})\rangle^\nu = 0$. The first non-vanishing sub-leading term in (40) is

$$\frac{m^{\nu+2}}{(\nu+2)!} \int d^2x_1 \cdots d^2x_n d^2x d^2y \langle \bar\psi\psi(x_1) \cdots \bar\psi\psi(x_n)\bar\psi\psi(x)\bar\psi\psi(y)\rangle^\nu. \tag{44}$$

This correlator is sometimes called a non-minimal correlator and was calculated in [30, 31, 68]. However, for our purpose we will only need the large distance limit of this correlator,

which again follows from the clustering property of the correlators. By decomposing the field into left handed and right handed components we find

$$\int d^2x_1...d^2x_\nu d^2x d^2y \langle\bar\psi\psi(x_1)\cdots\bar\psi\psi(x_\nu)\bar\psi\psi(x)\bar\psi\psi(y)\rangle_\nu \tag{45}$$
$$= (\nu+2)\int d^2x_1\cdots d^2x_\nu d^2x d^2y\langle\bar\psi_L\psi_R(x_1)\cdots\bar\psi_L\psi_R(x_\nu)\bar\psi_L\psi_R(x)\bar\psi_R\psi_L(y)\rangle^\nu.$$

The combinatorial factor $\nu+2$ arises because the fermion bilinear with opposite chiralities can be at each of the $\nu+2$ points. Using the clustering property for the expectation value we obtain for the large volume limit of (40)

$$\frac{m^{\nu+2}}{(\nu+2)!}(\nu+2)\Big(\frac{\Sigma}{2}\Big)^{\nu+2} = \frac{m^{\nu+2}}{(\nu+1)!}\Big(\frac{\Sigma}{2}\Big)^{\nu+2}. \tag{46}$$

The mass dependence of the partition function to this order is thus given by

$$\frac{Z^\nu(m)}{Z^0|_{m=0}} = \frac{m^\nu}{\nu!}\Big(\frac{\Sigma V}{2}\Big)^\nu + \frac{m^{\nu+2}}{(\nu+1)!}\Big(\frac{\Sigma V}{2}\Big)^{\nu+2} + O(m^{\nu+4}). \tag{47}$$

Comparing this result with (8), we arrive at the sum rule

$$\langle\langle\sum_{n\neq 0}\frac{1}{\lambda_n^2}\rangle\rangle^\nu = \frac{1}{|\nu|+1}\frac{\Sigma^2V^2}{2}, \tag{48}$$

which also incorporates the case of negative ν. In [20] this result was obtained from the mass dependence of the partition function at nonzero vacuum angle. In our derivation we only used the clustering property of expectation values. Therefore our derivation is also valid for QCD with one massless flavor. For the Schwinger model a derivation of the sum rule for zero topological charge was given in [26].

The term of the order $m^{\nu+2l}$, $l=2,3,...$ is not much more complicated. Again ν of the fermion bilinears are saturated by zero modes. The remaining ones are contracted by the nonzero mode Green's function, and therefore we have to choose l of the $\nu+2l$ fermion bilinears with opposite chirality [69],

$$\int d^2x_1\cdots d^2x_{\nu+2l}\langle\bar\psi\psi(x_1)\cdots\bar\psi\psi(x_{\nu+2l})\rangle^\nu \tag{49}$$
$$= \begin{pmatrix}\nu+2l\\ l\end{pmatrix}\int d^2x_1\cdots d^2x_{\nu+2l}\langle\bar\psi_L\psi_R(x_1)\cdots\bar\psi_L\psi_R(x_\nu)\bar\psi_L\psi_R(x_{\nu+1}\cdots\bar\psi_R\psi_L(x_{\nu+2l})\rangle^\nu.$$

Again using the clustering property we obtain for the large volume limit of the $m^{\nu+2l}$ order term in (40),

$$\frac{m^{\nu+2l}}{(\nu+2l)!}\frac{(\nu+2l)!}{(\nu+l)!l!}\Big(\frac{\Sigma V}{2}\Big)^{\nu+2l} = \frac{m^{\nu+2l}}{(\nu+l)!l!}\Big(\frac{\Sigma V}{2}\Big)^{\nu+2l}. \tag{50}$$

The term of order m^{2l} in the expansion of (7) is given by

$$m^{2l}\det{}'[i\not{D}]\sum_{\substack{n_i>0\\ n_1\neq\cdots\neq n_l}}\frac{1}{\lambda_{n_1}^2\cdots\lambda_{n_l}^2}. \tag{51}$$

Gathering all the pieces, we obtain the final result [20, 26]

$$\langle\langle\sum_{\substack{n_i\neq 0\\ n_1\neq\cdots\neq n_l}}\frac{1}{\lambda_{n_1}^2\cdots\lambda_{n_l}^2}\rangle\rangle^\nu=\frac{|\nu|!}{2^l\ l!\ (|\nu|+l)!}(\Sigma V)^{2l}. \tag{52}$$

This sum rule is valid for any theory with the above clustering property. In particular, it is valid for QCD with one flavor [20].

IV. MICROSCOPIC DERIVATION OF SUM RULES IN THE SCHWINGER MODEL

A microscopic derivation of the sum rule (48) in the sector of topological charge $\nu=0$ was given in [26]. In this section we generalize this derivation to arbitrary nonzero topological charge. The main idea of the derivation may be summarized in the following identity

$$\sum_{\lambda_n\neq 0}\frac{1}{\lambda_n^2}=-\mathrm{Tr}\Big[G^{\nu\,2}\Big], \tag{53}$$

where G^ν is the Green's function of the Dirac operator with external field $A_\mu=-\epsilon_{\mu\rho}\partial_\rho\phi$. The equality (53) follows from the spectral representation of the Green's function

$$G^\nu(x,y)=\sum_{n\neq 0}\frac{\psi_n^\phi(x)\psi_n^{\phi\,\dagger}(y)}{i\lambda_n}, \tag{54}$$

and the fact that excited Dirac states are normalized to 1 (the trace is both over Dirac indexes and spatial coordinates). Therefore, we need the Green's function $G^\nu(x,y)$ for an arbitrary background field ϕ in the sector of topological charge ν (which is taken $\nu>0$ for convenience). Due to the index theorem, the Dirac equation has exactly ν right-handed zero modes for a generic field configuration in this class. The Green's function $G^\nu(x,y)$ is thus defined by [70]

$$i\not{D}_x G^\nu(x,y)=\delta(x-y)-\gamma_5^+P^\nu(x,y), \tag{55}$$

where

$$P^\nu(x,y) = \sum_{p=1}^{\nu} \frac{\psi_p(x)\psi_p^\dagger(y)}{\int d^2z\psi_p^\dagger(z)\psi_p(z)}, \tag{56}$$

is the projector on the subspace of zero modes, $\psi_p(x),\ p = 1,\cdots,\nu$, and

$$\gamma_5^+ = \frac{1}{2}(1+\gamma_5). \tag{57}$$

In other words, $G_\nu(x,y)$ is the Green's function in the space of the nonzero modes. The explicit solution of (55) is given by:

$$G^\nu(x,y) = (1-\gamma_5^+ P^\nu)\tilde{G}^\nu(1-\gamma_5^+ P^\nu), \tag{58}$$

where, because $i\not{D}_\phi = e^{g\phi\gamma_5} i\not{\partial} e^{g\phi\gamma_5}$,

$$\tilde{G}^\nu(x,y) \equiv e^{-g\phi(x)\gamma_5} G_0(x-y) e^{-g\phi(y)\gamma_5}. \tag{59}$$

The free two-dimensional Dirac propagator $G_0(x,y)$ is given by

$$G_0(x,y) = \frac{1}{2\pi}\frac{\gamma_\mu(x_\mu - y_\mu)}{(x-y)^2}. \tag{60}$$

In the representation $\psi_1(x) = e^{-g\phi(x)}$ and $\{\psi_k, k = 2,\cdots,\nu\}$ is an orthogonal set perpendicular to ψ, one can easily show that (58) is a solution of (55). Using that $\gamma_5^+\tilde{G}^\nu\gamma_5^+ = 0$ we find the sum rule

$$\begin{aligned}\sum_{\lambda_n\neq 0}\frac{1}{\lambda_n^2} &= -\mathrm{Tr}\Big[(1-(1+\gamma_5)P^\nu)[G^\nu]^2\Big] \\ &= -\mathrm{Tr}\Big[(1+\gamma_5)(1-P^\nu)[G^\nu]^2\Big].\end{aligned} \tag{61}$$

In the second equality we have used that in the term proportional to the identity, both helicities give the same result. Using the explicit representation for the Green's function and carrying out the trace over the γ matrices, we find

$$\begin{aligned}\sum_{\lambda_n\neq 0}\frac{1}{\lambda_n^2} &= -\int d^2xd^2yd^2z(\delta(x-y) - P^\nu(x,y))tr\Big[(1+\gamma_5)e^{-g\phi(y)\gamma 5}G_0(y,z)e^{-2g\phi(z)\gamma_5}G_0(z,x)e^{-g\phi(x)\gamma_5}\Big] \\ &= -\frac{2}{(2\pi)^2}\int d^2xd^2yd^2z(\delta(x-y) - P^\nu(x,y))e^{-g\phi(y)-g\phi(x)+2g\phi(z)} \\ &\quad\times\frac{(x-z)_\mu(z-y)_\mu - i\epsilon_{\mu\nu}(y-z)_\mu(z-x)_\nu}{(x-z)^2(z-y)^2} \\ &= -\frac{1}{(2\pi)^2}\int d^2xd^2yd^2z(\delta(x-y) - P^\nu(x,y))e^{-g\phi(y)-g\phi(x)+2g\phi(z)} \\ &\quad\times\left[\frac{(x-y)^2 - 2i\epsilon_{\mu\nu}(y-z)_\mu(z-x)_\nu}{(x-z)^2(z-y)^2} - \frac{1}{(x-z)^2} - \frac{1}{(y-z)^2}\right].\end{aligned} \tag{62}$$

In the last equality we have used that

$$(x-z)_\mu (z-y)_\mu = \frac{1}{2}((x-y)^2 - (x-z)^2 - (z-y)^2). \tag{63}$$

The last two terms in the square brackets are independent of either y or x and vanish after integration as can be seen from the representation (56)

$$\int d^2y P^\nu(x,y) e^{-g\phi(y)} = e^{-g\phi(x)}. \tag{64}$$

Using that the integral over the δ-function also vanishes, we find the sum rule

$$\sum_{\lambda_n \neq 0} \frac{1}{\lambda_n^2} = \frac{1}{(2\pi)^2} \int d^2x d^2y d^2z P^\nu(x,y) e^{-g\phi(y)-g\phi(x)+2g\phi(z)} \frac{(x-y)^2 - 2i\epsilon_{\mu\nu}(y-z)_\mu (z-x)_\nu}{(x-z)^2(z-y)^2}. \tag{65}$$

The projector formula (56) is written in terms of orthogonal modes. However, since the zero modes (23) are not orthogonal, we require a projector formula that is valid for an arbitrary set of non-orthogonal modes. This is easily achieved by using a Lagrange interpolation such that the projector is equal unity for each of the zero modes. The required form is given by

$$P^\nu(x,y) = \frac{1}{(\nu-1)!} \frac{1}{\det \mathcal{N}} \sum_\sigma \mathrm{sgn}(\sigma) \begin{vmatrix} \psi_1^\dagger(y)\psi_{\sigma(1)}(x) & (\psi_1, \psi_{\sigma(2)}) & \cdots & (\psi_1, \psi_{\sigma(\nu)}) \\ \vdots & \vdots & & \vdots \\ \psi_\nu^\dagger(y)\psi_{\sigma(1)}(x) & (\psi_\nu, \psi_{\sigma(2)}) & \cdots & (\psi_\nu, \psi_{\sigma(\nu)}) \end{vmatrix}, \tag{66}$$

where the normalization factor is the familiar determinant of the norm matrix, given by (34). The projector can be expressed in terms of Slater determinants as

$$P^\nu(x,y) = \nu \frac{\int d^2x_2 \cdots d^2x_\nu \chi(x, x_2, \cdots, x_\nu) \chi^\dagger(y, x_2, \cdots, x_\nu)}{\int d^2x_1 \cdots d^2x_\nu |\chi(x_1, \cdots, x_\nu)|^2}. \tag{67}$$

Here, $\chi(x_1, \cdots, x_\nu)$ is the Slater determinant of fermionic zero modes. In this representation, it is particularly transparent that the projector is not only independent of the choice of basis, but (unlike $\det \mathcal{N}$), is also independent of the normalization of the zero modes, as of course it should be. Using the explicit representation of the fermionic zero modes we find the projector

$$\nu \frac{\int dx_2 \cdots dx_\nu \prod_{k=2}^\nu (x^+ - x_k^+)(y^- - x_k^-) \Delta(\{x_2^+, \cdots, x_\nu^+\}) \Delta(\{x_2^-, \cdots, x_\nu^-\}) e^{-g(\phi(x)+\phi(y)) - 2g(\phi(x_2)+\cdots+\phi(x_\nu))}}{\int dx_1 \cdots dx_\nu \Delta(\{x_1^+, \cdots, x_\nu^+\}) \Delta(\{x_1^-, \cdots, x_\nu^-\}) e^{-2g(\phi(x_1)+\cdots+\phi(x_\nu))}}, \tag{68}$$

where Δ is a Vandermonde determinant. Renaming x by x_1 and y by $x_{\nu+1}$, the sum rule in the sector of topological charge ν can be written as

$$\sum_{\lambda_n \neq 0} \frac{1}{\lambda_n^2} = \frac{2}{(2\pi)^{\nu+2}(\nu-1)!} \frac{1}{\det \mathcal{N}} \int d^2 z \int dx_1 \cdots dx_{\nu+1} e^{-2g(\phi(x_1)+\cdots+\phi(x_{\nu+1}))+2g\phi(z)}$$
$$\times \prod_{k=2}^{\nu} (x_1^+ - x_k^+)(x_{\nu+1}^- - x_k^-)\Delta(\{x_2^+, \cdots x_\nu^+\})\Delta(\{x_2^-, \cdots x_\nu^-\})$$
$$\times \frac{(x_1 - x_{\nu+1})^2 - 2i\epsilon_{\alpha\beta}(x_1 - z)_\alpha (z - x_{\nu+1})_\beta}{(x_1 - z)^2 (z - x_{\nu+1})^2}. \tag{69}$$

Only the symmetric part of the expression in the second line of this equation contributes to the integrand. Since the expression is already symmetric in $x_2, ..., x_\nu$, we only need to symmetrize with respect to x_1 and $x_{\nu+1}$. Then we get,

$$\mathcal{S} \prod_{k=2}^{\nu} (x_1^+ - x_k^+)(x_{\nu+1}^- - x_k^-)\Delta(\{x_2^+, \cdots x_\nu^+\})\Delta(\{x_2^-, \cdots x_\nu^-\})$$
$$\times \frac{(x_1 - x_{\nu+1})^2/2 - i\epsilon_{\mu\nu}(x_1 - z)_\mu (z - x_{\nu+1})_\nu}{(x_1 - z)^2 (z - x_{\nu+1})^2}$$
$$= -\frac{\Delta(\{x_1^+, \cdots x_{\nu+1}^+\})\Delta(\{x_1^-, \cdots x_{\nu+1}^-\})}{\nu(\nu+1)} \sum_{p \neq q} \frac{(x_p - x_q)^2/2 - i\epsilon_{\mu\nu}(x_p - z)_\mu (z - x_q)_\nu}{\prod_{l \neq p}(x_p^- - x_l^-) \prod_{l \neq q}(x_q^+ - x_l^+)(x_p - z)^2 (z - x_q)^2}. \tag{70}$$

The term with $p = q$ vanishes and can be included in the sum. Using that

$$\sum_p \frac{1}{\prod_{k \neq p}(x_p - x_k)} = 0, \tag{71}$$

we can replace the numerator

$$(x_p - x_q)^2/2 - i\epsilon_{\mu\nu}(x_p - z)_\mu (z - x_q)_\nu \tag{72}$$

in (70) by

$$(x_p - x_q)^2/2 - (x_p - z)^2/2 - (x_q - z)^2/2 - i\epsilon_{\mu\nu}(x_p - z)_\mu (z - x_q)_\nu$$
$$= -(x_p^+ - z^+)(x_q^- - z^-). \tag{73}$$

After simplifying the numerator we find for (70)

$$\frac{2\Delta(\{x_1^+, \cdots x_{\nu+1}^+\})\Delta(\{x_1^-, \cdots x_{\nu+1}^-\})}{\nu+1} \sum_{p,q} \frac{1}{\prod_{l \neq p}(x_p^- - x_l^-) \prod_{l \neq q}(x_q^+ - x_l^+)(x_p^- - z^-)(x_q^+ - z^+)}. \tag{74}$$

The sum in this equation factorizes into the product of two sums. Each sum can be expressed as a determinant

$$\sum_p \frac{\Delta(\{x_1^-,\cdots x_{\nu+1}^-\})}{\prod_{l\neq p}(x_p^- - x_l^-)(x_p^- - z^-)} = \begin{vmatrix} 1/(x_1^- - z^-) & 1 & x_1^- & \cdots x_1^{-\,\nu-1} \\ \vdots & & & \vdots \\ 1/(x_{\nu+1}^- - z^-) & 1 & x_{\nu-1}^- & \cdots x_{\nu+1}^{-\,\nu-1} \end{vmatrix}$$
$$= \frac{\Delta(\{x_1^-,\cdots x_{\nu+1}^-\})}{\prod_p(x_p^- - z^-)}. \tag{75}$$

The determinant is easily evaluated by multiplying each row by a factor $(x_k^- - z^-)$. Exactly the same simplification can be made for the $+$ variables. The sum rule therefore simplifies to

$$\sum_{\lambda_n\neq 0}\frac{1}{\lambda_n^2} = \frac{2}{(2\pi)^{\nu+2}(\nu+1)!}\frac{1}{\det\mathcal{N}}\int d^2z\int dx_1\cdots dx_{\nu+1}e^{-2g(\phi(x_1)+\cdots+\phi(x_{\nu+1}))+2g\phi(z)}$$
$$\times\frac{\Delta(\{x_1^+,\cdots x_{\nu+1}^+\})\Delta(\{x_1^-,\cdots x_{\nu+1}^-\})}{\prod_{q=1}^{\nu+1}(z^+ - x_q^+)(z^- - x_q^-)}. \tag{76}$$

This sum rule is valid for a arbitrary external gauge field $A_\mu = -\epsilon_{\mu\rho}\partial_\rho\phi$ in the sector of topological charge ν. The Leutwyler-Smilga sum rules (10) are obtained after averaging over the fields $\phi(x)$ with the effective action $\det\mathcal{N}e^{-\Gamma[\phi]}$, where $\Gamma[\phi]$ is defined in (29). Notice that effective action in the nontrivial topological sectors, due to the presence of $\det\mathcal{N}$, is not only non-Gaussian in ϕ but also non-local in ϕ. However, the factor $\det\mathcal{N}$ cancels in the average of the sum rule. The resulting path integral is Gaussian in ϕ and the average over the ϕ fields is given by a lowest order cumulant expansion. This results in

$$\left\langle\sum_{\lambda_n\neq 0}\frac{1}{\lambda_n^2}\right\rangle = \frac{2}{(2\pi)^{\nu+2}(\nu+1)!}\int d^2z\int dx_1\cdots dx_{\nu+1}e^{2(\nu+2)g^2G(0)+4g^2\sum_{k<l}^{\nu+1}G(x_k,x_l)-4g^2\sum_{k=1}^{\nu+1}G(x_k,z)}$$
$$\times\frac{\Delta(\{x_1^+,\cdots x_{\nu+1}^+\})\Delta(\{x_1^-,\cdots x_{\nu+1}^-\})}{\prod_{q=1}^{\nu+1}(z^+ - x_q^+)(z^- - x_q^-)}. \tag{77}$$

The second line of (77) is canceled by the asymptotic behavior of the exponentiated Green's functions in the first line of (77) (see (32)). The large volume limit of the sum rule is therefore given by

$$\left\langle\sum_{\lambda_n\neq 0}\frac{1}{\lambda_n^2}\right\rangle = \frac{2}{(2\pi)^{\nu+2}(\nu+1)!}V^{\nu+2}e^{2(\nu+2)g^2G(0)} = \frac{2}{(\nu+1)!}\left(\frac{\Sigma V}{2}\right)^{\nu+2}, \tag{78}$$

where we used (39). Now, using (43), and the definition (6), we finally obtain for the Leutwyler-Smilga sum rule,

$$\begin{aligned}\left\langle\left\langle\sum_{\lambda_n\neq 0}\frac{1}{\lambda_n^2}\right\rangle\right\rangle_\nu &= \frac{m^\nu Z^0(m)}{Z^\nu(m)}\bigg|_{m=0}\left\langle\sum_{\lambda_n\neq 0}\frac{1}{\lambda_n^2}\right\rangle \\ &= \frac{m^\nu Z^0(m)}{Z^\nu(m)}\bigg|_{m=0}\frac{2}{(2\pi)^{\nu+2}(\nu+1)!}V^{\nu+2}e^{2(\nu+2)g^2G(0)} = \frac{1}{2(\nu+1)}\Sigma^2V^2. \quad (79)\end{aligned}$$

The result (79) is in agreement with the sum rule obtained by Leutwyler and Smilga [20] and with the result obtained from Random Matrix Theory [24].

V. BOSONIZATION AND SUM RULES

In this section we derive sum rules for the inverse Dirac eigenvalues starting from a bosonic description of the massive Schwinger Model. In the particle physics literature, abelian bosonization goes back to the work of Lowenstein and Swieca [2], who essentially bosonized massless Schwinger Model, and Coleman [71] and Mandelstam [72]. The bosonized form of the massive Schwinger Model was studied in [73, 74]. Most earlier works used an operator language. The path integral approach to bosonization, which we will use below, was developed later, in many works, in particular in [75, 76].

It is well-known [73, 74] that the action of the massive Schwinger model can be written as

$$S[\varphi] = \int d^2x\Big[\frac{1}{2}(\partial_\nu\varphi)^2 + \frac{\mu^2}{2}\varphi^2 - cm\cos(2\sqrt{\pi}\varphi - \theta)\Big]. \quad (80)$$

with the constant $c = \mu e^{\gamma_E}/(2\pi) \equiv |\Sigma|$. The value of this constant is such that the chiral condensate is equal to its known value (24). At small m this is a theory of a weakly self-coupled massive scalar field. The physics of this model is discussed in detail in [73, 74]. In the mean field limit, where $\phi = 0$, we obtain (12) from the action (80). This directly leads to the Leutwyler-Smilga sum rules.

The first step in the microscopic derivation of sum rules is to eliminate the mass term of the φ field by

$$e^{-\int d^2x\frac{\mu^2}{2}\varphi^2} = \int D(\Delta\phi)e^{-\int d^2x[\frac{1}{2}(\Delta\phi)^2 + i\Delta\phi\varphi]}. \quad (81)$$

After partial integration of $(\partial\varphi)^2$ and shifting φ by $i\mu\phi$ we obtain the partition function

$$Z(m,\theta) = \int D\varphi D\phi e^{-S_{\rm eff}[\varphi,\phi]}, \tag{82}$$

with effective action given by

$$S_{\rm eff}[\phi,\varphi] = \int d^2x\Big[\frac{1}{2}\phi[\Delta^2 - \mu^2\Delta]\phi + \frac{1}{2}(\partial_\nu\varphi)^2 - \frac{1}{2}cm\Big[e^{2g\phi+i\ 2\sqrt{\pi}\varphi+i\theta} + e^{-2g\phi-i\ 2\sqrt{\pi}\varphi-i\theta}\Big]\Big]. \tag{83}$$

The integral $D\phi$ is only over modes with $\Delta\phi$ not identically equal zero. The partition function in the sector of topological charge ν is given by the Fourier transform of the partition function,

$$Z^\nu(m) = \int_0^{2\pi}\frac{d\theta}{2\pi}e^{i\theta\nu}Z(m,\theta). \tag{84}$$

The integral over θ is easily performed by expanding the integrand in powers of $e^{i\theta}$,

$$\begin{aligned} Z^\nu(m) = \int_0^{2\pi}\frac{d\theta}{2\pi}e^{i\theta\nu}\int D\varphi D\phi e^{-S_{\rm eff}(\varphi,\phi)}\sum_{k,l=0}^{\infty}\frac{(cm/2)^{k+l}}{k!l!}\int d^2x_1\cdots d^2x_k d^2y_1\cdots d^2y_l \\ \times\Big[\prod_{q=1}^{k}e^{-2g\phi(x_q)-i\ 2\sqrt{\pi}\varphi(x_q)-i\theta}\Big]\Big[\prod_{p=1}^{l}e^{2g\phi(y_p)+i\ 2\sqrt{\pi}\varphi(y_p)+i\theta}\Big]. \end{aligned} \tag{85}$$

It is clear that only the terms with $k = l+\nu$ contribute to the integral. However, we can also perform the integral after shifting

$$\phi \to \phi - \frac{i}{2g}\theta. \tag{86}$$

In the effective action this results in the extra term

$$i\theta\frac{g}{2\pi}\int d^2x\Delta\phi. \tag{87}$$

The integral over θ therefore vanishes unless

$$\frac{g}{2\pi}\int d^2x\Delta\phi = \nu, \tag{88}$$

i.e. the topological charge of the fields ϕ is equal to ν.

The path integral over φ is Gaussian, and is given by

$$\int D\varphi\Big[\prod_{p=1}^{l}e^{i\ 2\sqrt{\pi}\varphi(y_p)}\Big]\Big[\prod_{q=1}^{\nu+l}e^{-i\ 2\sqrt{\pi}\varphi(x_q)}\Big]e^{-S_0[\varphi]} = \frac{\prod_{k<r}^{l}(y_k-y_r)^2\prod_{i<j}^{\nu+l}(x_i-x_j)^2}{\prod_{q=1}^{\nu+l}\prod_{p=1}^{l}(y_p-x_q)^2}\int D\varphi e^{-S_0[\varphi]}. \tag{89}$$

with [89]

$$S_0[\varphi] = \frac{1}{2}\int d^2x(\partial_\mu\varphi)^2. \tag{90}$$

Thus we end up with

$$Z^\nu(m) = \int D\varphi e^{-S_0[\varphi]}\int D\phi e^{-\Gamma[\phi]}\sum_{l=0}^\infty \frac{(cm/2)^{2l+\nu}}{(\nu+l)!l!}\int d^2y_1\cdots d^2y_l d^2x_1\cdots d^2x_{\nu+l}$$
$$\times\frac{\prod_{k<r}^l(y_k-y_r)^2\prod_{i<j}^{\nu+l}(x_i-x_j)^2}{\prod_{q=1}^{\nu+l}\prod_{p=1}^l(y_p-x_q)^2}e^{2g[\phi(y_1)+\cdots+\phi(y_l)-\phi(x_1)-\cdots-\phi(x_{\nu+l})]}. \tag{91}$$

The path integral over ϕ can be performed by a leading order cumulant expansion. The Vandermonde determinants are again canceled by the large distance asymptotic behavior of the Green's functions. In the large volume limit we thus obtain

$$\frac{Z_\nu(m)}{Z_0(m=0)} = \sum_{l=0}^\infty\frac{m^{\nu+2l}}{(\nu+l)!l!}\Big(\frac{\Sigma V}{2}\Big)^{\nu+2l}, \tag{92}$$

which is equal to the result obtained from the clustering assumption The sum rules are the same as derived in section VI,

$$\langle\langle \sum_{\substack{n_i\neq 0\\ n_1\neq\cdots\neq n_l}}\frac{1}{\lambda_{n_1}^2\cdots\lambda_{n_l}^2}\rangle\rangle^\nu = \frac{|\nu|!}{2^l\ l!\ (|\nu|+l)!}(\Sigma V)^{2l}. \tag{93}$$

From this result one easily derives that $Z(m,\theta)=\exp(mV\Sigma\cos\theta)$ (see also [20, 69]).

We end this section with a conjecture for sum rules in a fixed external $\phi-$field. By now we realize that the field ϕ introduced in (81) is indeed the same as in previous sections. We can extract the sum rules from comparison with the partition function in eigenvalue representation. To that end, we rewrite the partition function as

$$Z^\nu(m) = m^\nu\int D\phi e^{-S[\phi]}\det{}'[i\not{D}]\frac{\det{}'[i\not{D}+m]}{\det{}'[i\not{D}]}$$
$$= \int D\phi e^{-\Gamma[\phi]}m^\nu\det\mathcal{N}^\nu\sum_{l=0}^\infty m^{2l}\sum_{\substack{n_i\neq 0\\ n_1\neq\cdots\neq n_l}}\frac{1}{\lambda_{n_1}^2\cdots\lambda_{n_l}^2}, \tag{94}$$

where the formulas (7) and (51) have been used. We remind that $\det\mathcal{N}^\nu$ is determinant of the norm matrix (34). We conjecture that the integrands of (91) and (94) are equal:

$$\sum_{\substack{n_i\neq 0\\ n_1\neq\cdots\neq n_l}}\frac{1}{\lambda_{n_1}^2\cdots\lambda_{n_l}^2} = \frac{\mathcal{C}}{(\nu+l)!l!\det\mathcal{N}^\nu}\int d^2y_1\cdots d^2y_l d^2x_1\cdots d^2x_{\nu+l} \tag{95}$$
$$\times\frac{\prod_{k<r}^l(y_k-y_r)^2\prod_{i<j}^{\nu+l}(x_i-x_j)^2}{\prod_{q=1}^{\nu+l}\prod_{p=1}^l(y_p-x_q)^2}e^{2g[\phi(y_1)+\cdots+\phi(y_l)-\phi(x_1)-\cdots-\phi(x_{\nu+1})]}.$$

For $l=1$ this conjecture was shown to be valid in in section IV.

VI. CLUSTERING FOR TWO OF MORE FLAVORS IN CHPT

In multi-flavor QCD, chiral symmetry is broken spontaneously, with the appearance of massless Goldstone bosons. This results in long range correlations that will modify the clustering property of scalar correlators and will affect the sum rules for the inverse Dirac eigenvalues. Starting from a chiral Lagrangian we will propose a modified clustering relation.

Let us first show that factorization leads to an incorrect result. To leading order in the quark masses the partition function in the sector of topological charge ν is given by

$$\frac{Z^\nu(m_1,\cdots,m_{N_f})}{Z^0(m_i=0)} = \frac{(m_1...m_{N_f})^\nu}{[\nu!]^{N_f}} \int \prod_{i,j} dx_i^{(j)} \Big\langle \bar\psi\psi(x_1^{(1)})\cdots\bar\psi\psi(x_\nu^{(1)})\cdots\bar\psi\psi(x_1^{(N_f)})\cdots\bar\psi\psi(x_\nu^{(N_f)})\Big\rangle^\nu . \tag{96}$$

If we assume clustering as in the case of one flavor we obtain

$$\Big\langle \bar\psi\psi(x_1^{(1)})\cdots\bar\psi\psi(x_\nu^{(1)})\cdots\bar\psi\psi(x_1^{(N_f)})\cdots\bar\psi\psi(x_\nu^{(N_f)})\Big\rangle^\nu \to \Big(\frac{\Sigma}{2}\Big)^{\nu N_f}, \tag{97}$$

resulting in the partition function

$$\frac{Z^\nu(m_1,\cdots,m_{N_f})}{Z^0(m_i=0)} = \frac{1}{[\nu!]^{N_f}} \left[\frac{m_1\Sigma V}{2}\right]^\nu \cdots \left[\frac{m_{N_f}\Sigma V}{2}\right]^\nu . \tag{98}$$

However, instead of $1/[\nu!]^{N_f}$, the correct pre-factor [20] is given by $\prod_{k=0}^{N_f-1} k!/(\nu+k)!$.

The reason for the incorrect pre-factor in (98) is that we did not account for the spontaneous symmetry breaking of $SU_A(N_f)$, which results in massless Goldstone bosons and long range correlations. As a consequence, the clustering property does not hold directly even for the correct ground state (i.e. at fixed θ). It is possible, as we will show below, to write down a modified clustering property.

The clustering property follows from the large distance behavior of the correlators. This is determined by the zero momentum term in the chiral Lagrangian, i.e. by

$$Z_{\rm Low}(M,\theta) = \mathcal{N}\int \mathcal{D}U e^{V\Sigma {\rm Re Tr}(MU^\dagger e^{i\theta/N_f})}, \tag{99}$$

with normalization constant $\mathcal{N}$ such that the partition function is equal to unity for $M=0$. For a diagonal mass matrix we obtain the expansion

$$Z_{\rm Low}(m_f,\theta) = \sum_{k_1,\cdots,k_{N_f}=0}^\infty \int_{U\in SU(N_f)} dU \prod_{l=1}^{N_f} \Big[\Big(\frac{m_l V\Sigma}{2}\Big)^{k_l} \frac{(U_{ll}^\dagger e^{i\theta/N_f} + U_{ll}e^{-i\theta/N_f})^{k_l}}{k_l!}\Big], \tag{100}$$

which should be compared to the large volume limit of the QCD partition function given by

$$Z^{\rm QCD}(m_f,\theta) = \sum_{k_1,\cdots,k_{N_f}=0}^{\infty} \prod_{l=1}^{N_f} \frac{(m_l V)^{k_l}}{k_l!} \langle \prod_{l=1}^{N_f} (\bar\psi_l \psi_l)_l^k \rangle_\theta. \tag{101}$$

We thus obtain the clustering relation

$$\langle \prod_{l=1}^{N_f} (\bar\psi_l \psi_l)^{k_l} \rangle_\theta = \int_{U\in SU(N_f)} dU \prod_{l=1}^{N_f} \frac{\Sigma}{2} (U_{ll}^\dagger e^{i\theta/N_f} + U_{ll} e^{-i\theta/N_f})^{k_l}. \tag{102}$$

If we isolate the phase ξ_i of the diagonal matrix element U_{ii} this can be rewritten more suggestively as

$$\langle \prod_{l=1}^{N_f} (\bar\psi_l \psi_l)^{k_l} \rangle_\theta = \int_{U\in SU(N_f)} dU \prod_{l=1}^{N_f} |U_{ll}| \langle \bar\psi\psi \rangle_{(\theta/N_f)-\xi_l}^{k_l}, \tag{103}$$

with $\langle \bar\psi\psi \rangle_{(\theta/N_f)-\xi_l}$ the chiral condensate for one flavor and vacuum angle equal to $(\theta/N_f)-\xi_l$. Notice that only the diagonal matrix elements of U enter in this expression. Since $|U_{ll}| \le 1$ we could interpret the factors $|U_{ll}|$ as the projection of the one flavor condensates onto certain directions in group space.

In general, the integral in (102) cannot be evaluated analytically. However, the situation simplifies enormously for the minimal correlators in the sector of fixed topological charge ν. In this case we have $n_1 = n_2 = \cdots = n_{N_f} = \nu$ and the integral over θ is only nonzero if no factors $U_{ll}^\dagger$ occur. This results in

$$\langle \prod_{l=1}^{N_f} (\bar\psi_l \psi_l)^{\nu} \rangle^\nu = \left(\frac{\Sigma}{2}\right)^{\nu N_f} \int_{U\in SU(N_f)} dU \prod_{l=1}^{N_f} U_{ll}^\nu. \tag{104}$$

This integral can be evaluated analytically (see appendix A)

$$\int dU U_{11}^\nu U_{22}^n \cdots U_{N_f N_f}^\nu = \left[\nu!\right]^{N_f} \prod_{i=0}^{N_f-1} \frac{i!}{(i+\nu)!}. \tag{105}$$

The leading mass dependence of the partition function is obtained by substituting the result (104) into equation (96),

$$Z^\nu(m_1,\cdots,m_{N_f}) = \left[\frac{m_1 \Sigma V}{2}\right]^\nu \cdots \left[\frac{m_{N_f}\Sigma V}{2}\right]^\nu \prod_{i=0}^{N_f-1} \frac{i!}{(i+\nu)!}. \tag{106}$$

This explains the suppression of the scalar correlator for $N_f \ge 2$.

VII. RELATIONS WITH RANDOM MATRIX THEORY

The chiral random matrix model in the sector of topological charge ν and N_f flavors with equal quark mass m is defined by [24, 25]

$$Z_{N_f}^{\nu}(m) = C_n^{(1)} \int dW \det{}^{N_f}(D+m) e^{-n\Sigma^2 \mathrm{Tr} W W^\dagger}, \tag{107}$$

where

$$D = \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix}, \tag{108}$$

and W is a complex $n \times (n+\nu)$ matrix. The eigenvalues of this model are distributed according to a semicircle. The largest eigenvalue is equal to $1/\Sigma$. We use this eigenvalue to normalize the partition function such that it becomes dimensionless. If we also normalize the partition function to unity for $N_f = 0$ we obtain the normalization constant

$$C_n^{(1)} = e^{nN_f} \left(\frac{n}{\pi}\right)^{n(n+\nu)} \Sigma^{2n(n+\nu)}. \tag{109}$$

We also have included the factor e^{nN_f} which eliminates the constant vacuum energy for $N_f \neq 0$. We can evaluate the partition function (107) in the limit $m \to 0$ in two different ways. First, using an eigenvalue representation of W, and second, using a σ-model representation of the large n limit of the partition function. Since the σ-model is exactly the partition function (99), we only discuss the first approach in the next subsection.

A. Eigenvalue Representation

To obtain an eigenvalue representation of the partition function we use the polar decomposition

$$W = U \Lambda V^{-1}, \tag{110}$$

where $U \in U(n+\nu)/U^n(1) \times U(\nu)$ and $V \in U(n)$. The Jacobian of this transformation is given by

$$J = \Delta(\{\lambda_k^2\}) 2^n \prod_{k=1}^{n} (\lambda_k)^{2\nu+1}. \tag{111}$$

The eigenvalues of Λ are positive or zero, and the integration is over an ordered sequence of eigenvalues which will be accounted for by a factor $1/n!$ below. To lowest non-vanishing order in m, the eigenvalue representation of the partition function is thus given by

$$Z^{\nu}_{N_f}(m) = C^{(1)}_n C^{(2)}_n m^{\nu N_f} \frac{1}{n!} \int \prod_{k=1}^{n} 2 d\lambda_k \lambda_k^{2(\nu+N_f)+1} \Delta^2(\{\lambda_k^2\}) e^{-n\Sigma^2 \sum_k \lambda_k^2}, \tag{112}$$

where we have introduced a second constant,

$$C^{(2)}_n = 2^{-n^2-n\nu} \frac{\mathrm{vol}\,(U(n+\nu))\mathrm{vol}\,(U(n))}{\mathrm{vol}\,(U(\nu))\mathrm{vol}^n\,(U(1))}, \tag{113}$$

and the volume of $U(n)$ is given by

$$\mathrm{vol}\,(U(n)) = \frac{(2\pi)^{n(n+1)/2}}{\prod_{k=0}^{n-1} k!}. \tag{114}$$

We rewrite the partition function in terms of the integration variables

$$x_k = 2n\Sigma^2\lambda_k^2. \tag{115}$$

This results in

$$Z^{\nu}_{N_f}(m) = C^{(3)}_n C^{(2)}_n m^{\nu N_f} \frac{1}{n!} \int \prod_{k=1}^{n} dx_k x_k^{\nu+N_f} \Delta^2(\{x_k\}) e^{-x_k/2}, \tag{116}$$

where

$$C^{(3)}_n = (2n)^{-[n(n+\nu)+nN_f]} \pi^{-n(n+\nu)} e^{nN_f}. \tag{117}$$

The integral over the x_k was calculated by Forrester [77]. Our final result for the partition function is thus given by

$$Z^{\nu}_{N_f}(m) = m^{\nu N_f} \Sigma^{-2nN_f} C^{(3)}_n C^{(2)}_n 2^{n(\nu+N_f+n)} \prod_{j=0}^{n-1} j!(\nu+N_f+j)!. \tag{118}$$

Collecting all factors we find the partition function

$$Z^{\nu}_{N_f}(m) = m^{\nu N_f} \Sigma^{-2nN_f} n^{-nN_f} e^{nN_f} \prod_{j=0}^{n-1} \frac{(\nu+N_f+j)!}{(\nu+j)!}. \tag{119}$$

Using the Stirling formula, we find in the thermodynamic limit

$$Z^{\nu}_{N_f}(m) = m^{\nu N_f} \Sigma^{-2nN_f} \frac{(2\pi)^{N_f/2} n^{N_f(N_f+2\nu)/2}}{\prod_{k=0}^{N_f-1} (\nu+k)!}. \tag{120}$$

For $N_f = 1$ we find the ratio

$$\frac{Z^{\nu=1}_{N_f=1}(m)}{Z^{\nu=0}_{N_f=1}(m)} = m(n+1). \tag{121}$$

According to (38) the condensate is given by

$$\lim_{V\to\infty}\lim_{m\to 0}\frac{1}{mV}\left[\frac{Z^{\nu=1}_{N_f=1}(m)}{Z^{\nu=0}_{N_f=1}(m)} + \frac{Z^{\nu=-1}_{N_f=1}(m)}{Z^{\nu=0}_{N_f=1}(m)}\right]. \tag{122}$$

In random matrix theory the volume is identified as $V = 2n$ such that (121) results in a condensate equal to unity. The reason for this incorrect result is that we have evaluated the ratio of two products with the same number of nonzero eigenvalues. To obtain a result with the correct dimensionality we have to regularize the determinants by only including eigenvalues below a given energy [90]. For $\nu = 1$ we have on average one eigenvalue less with absolute value below a given energy than for $\nu = 0$. This results in an overall factor of $1/p\Delta$, where p is the number of eigenvalues below the cut-off energy and Δ is the spacing between the eigenvalues. Instead of using a fixed energy cut-off, we calculate the ratio of the partition functions for a fixed total number of eigenvalues meaning that in the sector of topological charge ν we make the replacement $n \to n - \nu/2$ in (120). We finally find the ratio

$$\frac{Z^{\nu}_{N_f}(m)}{Z^{0}_{N_f}(m=0)} = (mn\Sigma)^{\nu N_f}\prod_{k=0}^{N_f-1}\frac{k!}{(k+\nu)!}, \tag{123}$$

in agreement with general arguments given in [20]. This result also follows from the fact that the large N limit of the random matrix model is given by the nonlinear sigma model (99) [24]. As was already found in [24], the result (123) confirms that the chiral anomaly is contained in random matrix theory.

Since the finite volume partition function for $N_f = 1$ is a Bessel function, it is tempting to identify its zeros with the average position of the eigenvalues. In the random matrix model it can be easily checked whether this identification is justified. Numerically, it turns out that the ratio obtained from the average eigenvalues is a factor $2/\pi$ smaller than the result given in (121). A second question we have asked is whether the average determinant is mainly given by the product of the average eigenvalues or whether it is

due to the fluctuations of the eigenvalues. It turns out that the product of the average eigenvalues (with the determinant included in the weight) is an order of magnitude larger than the average determinant.

VIII. DISCUSSION

A. Spectral Duality

The chiral condensate can be obtained in two ways. First, by taking the chiral limit after the thermodynamic limit,

$$\Sigma^{(1)} = \lim_{m\to 0}\lim_{V\to\infty}\frac{1}{V}\left\langle\frac{1}{i\lambda_k + m}\right\rangle, \tag{124}$$

and, second, by inverting the two limits,

$$\Sigma^{(2)} = \lim_{V\to\infty}\lim_{m\to 0}\frac{1}{V}\left\langle\frac{1}{i\lambda_k + m}\right\rangle. \tag{125}$$

In the first case, the condensate arises as a consequence of spontaneous symmetry breaking, and in the second case it is due to the anomaly or instantons. The two condensates are not necessarily equal. For example, if we restrict the partition function to gauge field configurations with zero topological charge, the second definition gives zero. By spectral duality we mean that the two condensates are equal for the θ - vacuum [79]. In the second case, when the contribution to the condensate comes from the sector with topological charge equal to one, the eigenvalues for $\nu = 1$ are shifted such that the condensate is equal to the one obtained from the first definition. The first definition implies the Banks-Casher formula [66]

$$\Sigma^{(1)} = \frac{\pi\rho_0}{V}. \tag{126}$$

This implies that the average spacing of the eigenvalues is given by

$$\Delta = \frac{1}{\rho(0)} = \frac{\pi}{\Sigma V}. \tag{127}$$

If we assume that the position of the eigenvalue λ_n is given by $n\Delta$, we obtain the sum rule

$$\sum_{n>0}\frac{1}{n^2\Delta^2} = \frac{1}{6}\Sigma^2 V^2, \tag{128}$$

which explains the functional dependence of the sum rule.

As we will show next, the large ν limit of the sum rule can be obtained from the large ν limit of the average position of the eigenvalues. In the sector of topological charge ν the "average" position of the eigenvalues can be expressed in terms of the zeros, $j_{\nu,k}$, of the Bessel functions [20],

$$\lambda_k = \pm \frac{j_{\nu,k}}{\Sigma V}. \tag{129}$$

For large k at fixed ν one one finds

$$\lambda_k \sim j^{\rm as}_{\nu,k}\Delta \sim (k + \frac{\nu}{2} - \delta)\Delta \qquad \text{with} \qquad \delta = \frac{1}{4}. \tag{130}$$

Here, $\Delta \equiv 1/\rho(0)$ is the average spacing of the eigenvalues. This formula does not accurately give the position of the eigenvalues for large ν and finite k. In this case one can use a uniform asymptotic expansion of the zeros given by

$$j_{\nu,k} \sim \nu z_k, \tag{131}$$

with z_k implicitly defined by $(4k-1)\pi = 4\nu(\sqrt{z_k^2-1} - \arccos(1/z_k))$. This results in the correct large ν limit of the sum rule

$$\sum_k \frac{1}{\lambda_k^2} \sim \Sigma^2 V^2 \int_0^\infty dk \frac{1}{j^2_{\nu,k}} = \frac{\Sigma^2 V^2}{\pi\nu} \int_1^\infty dz \frac{\sqrt{z^2-1}}{z^3} = \frac{\Sigma^2 V^2}{4\nu}. \tag{132}$$

Let us now calculate the condensate according to the second definition

$$\Sigma^{(2)} = \frac{1}{V} \frac{\langle \prod'_k (i\lambda_k + m)\rangle_{\nu=1}}{\langle \prod_k (i\lambda_k + m)\rangle_{\nu=0}} + \frac{1}{V} \frac{\langle \prod'_k (i\lambda_k + m)\rangle_{\nu=-1}}{\langle \prod_k (i\lambda_k + m)\rangle_{\nu=0}}. \tag{133}$$

Using the expression (13) for the partition function, this can be rewritten as

$$\Sigma^{(2)} = \lim_{m\to 0} \frac{1}{V} \left[\frac{I_1(mV\Sigma)/m}{I_0(mV\Sigma)} + \frac{I_{-1}(mV\Sigma)/m}{I_0(mV\Sigma)} \right] = \Sigma. \tag{134}$$

The ratio in (133) can be interpreted as the ratio of the expectation values of the fermion determinant in the topological sector $\nu = 1$ and $\nu = 0$.

We will calculate the averages in (133) by replacing the eigenvalues according to (129)

$$\begin{aligned} \lim_{m\to 0} \frac{1}{V} \frac{\langle \prod'_k (i\lambda_k + m)\rangle_{\nu=1}}{\langle \prod_k (i\lambda_k + m)\rangle_{\nu=0}} &\approx \lim_{m\to 0} \frac{1}{V} \frac{\prod'_k (ij_{\nu=1,k}/(\Sigma V) + m)}{\prod_k (ij_{\nu=0,k}/(\Sigma V) + m)} \\ &= \frac{1}{V} \frac{\prod'_k ij_{\nu=1,k}\Delta/\pi}{\prod'_k ij^{as}_{\nu=1,k}\Delta/\pi} \frac{\prod_k ij^{\rm as}_{\nu=0,k}\Delta/\pi}{\prod_k ij_{\nu=0,k}\Delta/\pi} \frac{\prod'_k ij^{\rm as}_{\nu=1,k}\Delta/\pi}{\prod_k ij^{\rm as}_{\nu=0,k}\Delta/\pi}. \end{aligned} \tag{135}$$

The first two ratios are finite, but the third ratio has to be regularized which we will do by a ζ-function regularization (see Appendix B). For $m = 0$ we find the result (see (183)

$$\frac{1}{V}\frac{\prod_k' ij^{\rm as}_{\nu=1,k}\Delta/\pi}{\prod_k ij^{\rm as}_{\nu=0,k}\Delta/\pi} = \frac{1}{V\Delta}\frac{\Gamma^2(1-\delta)}{\Gamma^2(\frac{3}{2}-\delta)} = \frac{\Sigma^{(1)}}{\pi}\frac{\Gamma^2(1-\delta)}{\Gamma^2(\frac{3}{2}-\delta)}. \tag{136}$$

The ratios $\prod_k' ij_{\nu,k}/\prod_k' ij^{as}_{\nu=1,k}$ can be easily evaluated numerically. We find

$$\frac{\prod_k ij_{\nu=0,k}}{\prod_k ij^{as}_{\nu=0,k}} = \Gamma^2(\frac{3}{4})/\sqrt{2}, \qquad \frac{\prod_k' ij_{\nu=1,k}}{\prod_k' ij^{as}_{\nu=1,k}} = \pi\Gamma^2(1.25)/2\sqrt{2}. \tag{137}$$

Therefore,

$$\Sigma^{(2)} = \Sigma^{(1)}. \tag{138}$$

Another way of evaluating the ratio (136) is from the limit

$$\lim_{n\to\infty}\frac{1}{n}\frac{\prod_{k=-n}^{n}{}' ij^{\rm as}_{\nu=1,k}\Delta/\pi}{\prod_{k=-n}^{n} ij^{\rm as}_{\nu=0,k}\Delta/\pi} = \lim_{n\to\infty}\frac{1}{n}\left[\frac{\prod_{k=0}^{n-1}(k+\frac{5}{4})}{\prod_{k=0}^{n-1}(k+\frac{3}{4})}\right]^{1/2} \tag{139}$$

Using the infinite product representation of the Γ-function we find that this limit is given by $\Gamma^2(\frac{3}{4})/\Gamma^2(\frac{5}{4})$. We thus have

$$\left.\frac{\prod_k' ij^{\rm as}_{\nu=1,k}\Delta/\pi}{\prod_k ij^{\rm as}_{\nu=0,k}\Delta/\pi}\right|_\zeta = \frac{1}{\Delta}\lim_{n\to\infty}\frac{1}{n}\frac{\prod_{k=0}^{n}{}' ij^{\rm as}_{\nu=1,k}\Delta/\pi}{\prod_{k=0}^{n} ij^{\rm as}_{\nu=0,k}\Delta/\pi}, \tag{140}$$

so that the definition (139) of the infinite product would have resulted in an incorrect value for the chiral condensate. This explains why in the random matrix calculation of previous section the incorrect chiral condensate is obtained from (121). The fact that the two results differ by the eigenvalue at the cut-off is in agreement with the interpretation that ζ-function regularization includes the product of the eigenvalues up to a fixed energy.

B. Random Gauge Field

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In the condensed matter literature Dirac fermions in random gauge fields have received a considerable amount of attention. Among the different types of models that have been considered, we mention the random magnetic field problem [80], the random flux model [35], the random mass model [34] and the random gauge field model [34]. In this section

we only consider the latter model which is the quenched Schwinger model with the gauge field action is replaced by

$$\frac{1}{4}F_{\mu\nu}^2 \to \frac{1}{2}\frac{g^2}{\sigma^2}A_\mu A_\mu. \tag{141}$$

The factor g^2 is introduced to have the same normalization as in [34]. Also in this case we can make a Hodge decomposition of the gauge field [81],

$$A_\mu = \epsilon_{\mu\nu}\partial_\nu\phi + \partial_\mu\Lambda. \tag{142}$$

Let us consider the action of an instanton configuration given by

$$\phi = \frac{1}{2g}\log(x^2+\rho^2). \tag{143}$$

In the Schwinger model the action of this configuration is given by

$$S_{\rm inst} = \frac{1}{2}\int d^2x F_{01}^2 = \frac{1}{2}\int d^2x \partial^2\phi\partial^2\phi = \frac{\pi}{3g^2\rho^2}, \tag{144}$$

which is infrared finite. Also the anomalous contribution to the effective action in the Schwinger model is infrared finite,

$$-\int d^2x \frac{\mu^2}{2}\phi\partial^2\phi = \frac{1}{2}(\log\rho^2 + 1). \tag{145}$$

For the random gauge model, on the other hand, we obtain the action

$$\begin{aligned}\frac{1}{2}\frac{g^2}{\sigma^2}\int_{|x|<R} d^2x A_\mu^2 &= \frac{1}{2}\frac{g^2}{\sigma^2}\int_{|x|<R} d^2x \partial_\mu\phi\partial_\mu\phi \\ &= \frac{\pi}{2\sigma^2}\Big[\log(\frac{R^2+\rho^2}{\rho^2}) - \frac{R^2}{R^2+\rho^2}\Big].\end{aligned} \tag{146}$$

which diverges for $R\to\infty$ which implies that instantons are suppressed in the thermodynamic limit. Therefore the chiral condensate of the massless theory is zero, and by spectral duality, the eigenvalue density of the Dirac operator vanishes independent of the topological charge. For $\sigma^2 = \pi$ the suppression factor is $1/V$, which exactly as in the Schwinger model for $N_f = 1$, results in a finite spectral density for $E\to 0$. Notice that the difference between the anomalous contribution in effective action of the Schwinger model and the random gauge action is only in the boundary term, which diverges for $R\to\infty$.

Let us evaluate the spectral density of a random gauge model with N_f massless flavors and n flavors with mass m. Because of the above remark, we restrict ourselves to the trivial topological charge sector, where the boundary term that gives the diverging result in (146) is absent. Using that the pure gauge term decouples from the partition function, the effective action of this model is given by

$$S_{\rm RG} = \int d^2x \Big[-\frac{1}{2}\phi[\frac{g^2}{\sigma^2}\Delta + \mu^2(N_f+n)\Delta]\phi - i\frac{\theta g}{2\pi}\Delta\phi + \sum_\alpha \bar\psi_\alpha[i\not\partial + me^{2g\gamma_5\phi}]\psi_\alpha\Big]. \tag{147}$$

The massless fermion fields result in an overall constant which can be ignored. The action of the massive fermion fields will be replaced by their bosonized form given by

$$S_{\rm F}(\phi,\varphi_\alpha) = \frac{1}{2}\sum_{\alpha=1}^n (\partial_\mu\varphi_\alpha)^2 - cm\sum_{\alpha=1}^n \cos(2\varphi_\alpha\sqrt{\pi} - 2ig\phi). \tag{148}$$

If we define

$$\Gamma(\phi) = -\frac{1}{2}\phi(\frac{g^2}{\sigma^2}\nabla^2 + \mu^2(N_f+n)\nabla^2)\phi - i\frac{\theta g}{2\pi}\Delta\phi, \tag{149}$$

the total bosonic action is given by

$$S_{\rm RG} = \int d^2x[\Gamma(\phi) + S_F(\phi,\varphi_\alpha)]. \tag{150}$$

After shifting the φ_α fields by

$$\varphi_\alpha \to \varphi_\alpha + i\mu\phi - \frac{\theta}{2\sqrt{\pi}}, \tag{151}$$

we obtain the action density

$$\frac{1}{2}\sum_{\alpha=0}^n (\partial_\mu\varphi_\alpha)^2 + \frac{1}{2}(\frac{g^2}{\sigma^2} + \mu^2 N_f)(\partial_\mu\phi)^2 - cm\sum_{\alpha=1}^n \cos(2\varphi_\alpha\sqrt{\pi} - \theta) - i\mu\partial_\mu\phi\sum_{\alpha=1}^n \partial_\mu\varphi_\alpha. \tag{152}$$

The final action density is obtained after performing the Gaussian integral over ϕ,

$$\frac{1}{2}\zeta\sum_{\alpha=1}^n\sum_{\beta=1}^n \partial_\mu\varphi_\alpha\partial_\mu\varphi_\beta + \frac{1}{2}\sum_{\alpha=1}^n(\partial_\mu\varphi_\alpha)^2 - cm\sum_{\alpha=1}^n \cos(2\varphi_\alpha\sqrt{\pi} - \theta), \tag{153}$$

where

$$\zeta = \frac{\mu^2}{\mu^2 N_f + g^2/\sigma^2}. \tag{154}$$

Considering the cosine term as a vertex, the propagator of the φ_α fields is given by

$$G_{\alpha\beta}(p) = \frac{1}{p^2}(\delta_{\alpha\beta} - \frac{\zeta}{1+n\zeta}). \tag{155}$$

Using this result one can easily derive the dependence of the chiral condensate on the infrared cutoff which is taken to be equal to the size of the box. Following the derivation of [37] and using the lattice spacing a as ultraviolet cutoff we find for $a/L \ll 1$ that

$$G_{\alpha\beta}(|x| \to 0) = -\frac{1}{2\pi}(\delta_{\alpha\beta} - \frac{\zeta}{1+n\zeta}) \log \frac{a}{L} \tag{156}$$

The chiral condensate is given by

$$\begin{aligned} \Sigma &= \frac{1}{n}\frac{1}{V}\partial_m \log Z \sim \langle e^{2i\sqrt{\pi}\varphi_1} \rangle \\ &= e^{-2\pi G_{11}(|x|\to 0)} \sim \left(\frac{a}{L}\right)^{1-\zeta/(1+n\zeta)} \end{aligned} \tag{157}$$

In the replica limit, $n \to 0$, we obtain the condensate

$$\Sigma \sim \frac{1}{L^{1-\zeta}}, \tag{158}$$

According to the Banks-Casher the smallest nonzero eigenvalue can be estimated as

$$\lambda_{\rm min} \sim \frac{\pi}{\Sigma V} \sim \frac{1}{L^{1+\zeta}}. \tag{159}$$

On the other hand, if the eigenvalue density is given by

$$\rho(\lambda) \sim V\lambda^\alpha, \tag{160}$$

the smallest nonzero eigenvalue follows from

$$\int_0^{\lambda_{\rm min}} \rho(\lambda) d\lambda \approx 1. \tag{161}$$

Therefore,

$$\lambda_{\rm min} \sim \frac{1}{L^{2/(\alpha+1)}}, \tag{162}$$

so that

$$\alpha = \frac{1-\zeta}{1+\zeta}. \tag{163}$$

This result has also been derived from a more sophisticated renormalization group analysis [34, 82, 83]. For the action (153) we obtain

$$\alpha = \frac{g^2 + \mu^2\sigma^2(N_f-1)}{g^2 + \mu^2\sigma^2(N_f+1)} = \frac{1+\sigma^2(N_f-1)/\pi}{1+\sigma^2(N_f+1)/\pi}. \tag{164}$$

In the quenched case ($N_f = 0$), this result agrees with the result obtained in [34], and in the limit $\sigma \to \infty$ we recover the result for the N_f flavor Schwinger model [26, 65]. For $N_f = 1$ the spectral density is qualitatively different from the Schwinger model. A diverging spectral density for $N_f = 0$ has been observed in recent lattice simulations [18]. If the spectral density vanishes, it still possible to derive sum rules for the inverse Dirac eigenvalues [84, 85, 86, 87]. It could be that such sum rules are easier obtained for lattice simulation of random Dirac fermions than for lattice simulations of the Schwinger model with two massless flavors [18].

A special case where the sum rules for the inverse Dirac eigenvalues can be evaluated is $N_f = 0$ and $n = 1$. Since the sum rule follows from the coefficient of m^2 in the expansion of the partition function in powers of m, the spectral density corresponding to this sum rule is the one of the Schwinger model with one massless flavor.

For $n = 1$ and $N_f = 0$ the partition function after the transformation $\varphi_1 \to \mu\phi$ is given by

$$Z(m, \theta) = \int D\varphi e^{\frac{1}{2}F^2 \int d^2x \phi\Delta\phi + cm\frac{1}{2}\int d^2x [e^{-i\theta+2ig\phi(x)} + e^{i\theta-2ig\phi(x)}]}, \tag{165}$$

where

$$F^2 = \mu^2 + \frac{\mu^2\sigma^2}{\pi}. \tag{166}$$

Let us evaluate the simplest sum rule in the sector of zero topological charge. To this end we expand the partition function to second order in m and integrate over θ. We obtain

$$\partial_m^2 \log Z^{\nu=0}(m) = \int d^2x d^2y e^{4g^2 G^{\rm RG}(x,y) - 4g^2 G^{\rm RG}(0,0)}, \tag{167}$$

where the random gauge field Green's function is given by

$$G^{\rm RG}(x, y) = -\frac{1}{4\pi F^2} \log(x - y)^2. \tag{168}$$

The volume dependence of the sum rule is thus given by

$$\partial_m^2 \log Z^{\nu=0}(m) \sim V^{2-\mu^2/F^2}. \tag{169}$$

If the eigenvalue density scales as

$$\rho(E) \sim V E^{\alpha}, \tag{170}$$

the volume dependence of the sum rule is given by

$$\sum_k \frac{1}{\lambda_k^2} \sim V^{\frac{2}{1+\alpha}}, \tag{171}$$

so that the exponent α is equal to

$$\alpha = \frac{\mu^2}{2F^2 - \mu^2} = \frac{1}{1 + 2\sigma^2/\pi}. \tag{172}$$

Indeed, this result agrees with (164) for $N_f = 1$.

IX. CONCLUSIONS

In conclusion, we have analyzed Leutwyler-Smilga sum rules in the Schwinger model and found that they are in agreement with the universal result from chiral perturbation theory and random matrix theory. Instead of relying on general symmetry arguments, we have performed a microscopic calculation of the sum of the inverse square Dirac eigenvalues in the sector of arbitrary topological charge ν, generalizing a result by Smilga obtained in the sector of zero topological charge.

We have shown that validity of the sum rules follows from the clustering property of the scalar correlation functions. This argument also applies to QCD with one flavor. For QCD with several flavors, the naive clustering argument fails due to the presence of Goldstone bosons. A modified clustering property is obtained from the chiral Lagrangian that corresponds to the low energy limit of QCD partition function.

The dependence of the sum rules on the topological charge ν is consistent with a shift of the Dirac eigenvalues by $\nu/2$ times the average level spacing. Such shift of the Dirac spectrum exactly results in the chiral condensate for massless quarks in the sector of topological charge $\nu = 1$. However, obtaining this result, requires a regularization of the fermion determinant with a fixed energy cutoff such as for example the ζ-function regularization.

In the microscopic derivation, the sum rules where obtained from more general sum rules valid for a fixed external gauge field with topological charge ν. The most general sum rules in this class were conjectured based a simplified derivation of Leutwyler-Smilga sum rules starting from the bosonized Schwinger model. It would be interesting to probe such sum rules directly in lattice simulations.

The Schwinger model is part of a larger class of models known under the name of random Dirac fermions which have been investigated in the context of the quantum Hall effect. The key difference between the two models is that instantons are suppressed by the random gauge field action. Therefore, the chiral condensate of the massless theory vanishes in the domain below a critical value of the disorder. In these models we expect modified sum rules that would be a useful probe for the scaling behavior of the density of states near zero.

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X. APPENDIX A

In this appendix we calculate the integral given in (105). We start from the result [88],

$$W(J) = \int_{SU(N_f)} dU \exp(Tr(JU)) \\ = \sum_{i=0}^{\infty} \frac{2! \cdots (N_f-1)!}{i! \cdots (i+N_f-1)!} \Big(\det J \Big)^i. \tag{173}$$

We are interested in the integral

$$\int_{SU(N_f)} dU U_{11}^\nu U_{22}^\nu \cdots U_{N_f N_f}^\nu = \left[\Big(\frac{d}{dJ_{11}} \Big)^\nu \cdots \Big(\frac{d}{dJ_{N_f N_f}} \Big)^\nu W(J) \right] \Big|_{J=0}. \tag{174}$$

On the other hand, we can choose $J = \mathrm{diag}[J_{11}, J_{22}, \cdots, J_{N_f N_f}]$, so that

$$\det J = J_{11} J_{22} \cdots J_{N_f N_f}. \tag{175}$$

Using this result, we arrive at N_f independent differentiations over diagonal matrix elements of J. The only term in the sum that contributes is the one with $i = \nu$ resulting in

$$\int_{SU(N_f)} dU U_{11}^\nu U_{22}^n \cdots U_{N_f N_f}^\nu = \Big[\nu! \Big]^{N_f} \frac{2! \cdots (N_f-1)!}{\nu! \cdots (\nu+N_f-1)!} \\ = \Big[\nu! \Big]^{N_f} \prod_{i=0}^{N_f-1} \frac{i!}{(i+\nu)!}. \tag{176}$$

XI. APPENDIX B

In this appendix we compute the determinant of the Dirac operator in ζ-function regularization with eigenvalues given by the asymptotic values of the zeros of Bessel functions,

$$j^{\rm as}_{\nu,k} = k + \frac{\nu}{2} - \delta \qquad \text{with} \qquad \delta = \frac{1}{4}. \tag{177}$$

In a ζ function regularization the determinant of the Dirac operator in the sector of topological charge ν is given by

$$\log \prod_k{}' (i\Delta j^{\rm as}_{\nu,k}/\pi + m) = -\left.\frac{d}{ds}\right|_{s=0} \left[\sum_{k=1}^{\infty} \frac{1}{(k+\frac{\nu}{2}-\delta+im/\Delta)^s\Delta^s} + \sum_{k=1}^{\infty} \frac{1}{(k+\frac{\nu}{2}-\delta-im/\Delta)^s\Delta^s} \right]. \tag{178}$$

The function in between the brackets is known as the Hurwitz zeta function,

$$\zeta(s,a) \equiv \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}. \tag{179}$$

For convenience we only calculate the product for $m = 0$. For $\nu = 1$ we find

$$\begin{aligned} \log \prod_k{}' i\frac{\Delta}{\pi} j^{\rm as}_{\nu=1,k} &= -2 \left.\frac{d}{ds}\right|_{s=0} [\zeta(s, \frac{3}{2}-\delta)\Delta^{-s}] \\ &= -2\zeta'(0, \frac{3}{2}-\delta) + 2\zeta(0, \frac{3}{2}-\delta)\log\Delta. \end{aligned} \tag{180}$$

Combining this with the result for $\nu = 0$,

$$\begin{aligned} \log \prod_k{}' i\frac{\Delta}{\pi} j^{\rm as}_{\nu=0,k} &= -2 \left.\frac{d}{ds}\right|_{s=0} [\zeta(s, 1-\delta)\Delta^{-s}] \\ &= -2\zeta'(0, 1-\delta) + 2\zeta(0, 1-\delta)\log\Delta, \end{aligned} \tag{181}$$

and using that

$$\zeta(0,a) = \frac{1}{2} - a, \qquad \zeta'(0,a) = \log\Gamma(-\frac{1}{2}\log(2\pi), \tag{182}$$

we find the ratio,

$$\frac{1}{V}\frac{\prod_k' i\frac{\Delta}{\pi}j^{\rm as}_{\nu=1,k}}{\prod_k i\frac{\Delta}{\pi}j^{\rm as}_{\nu=0,k}} = \frac{1}{V\Delta}\frac{\Gamma^2(1-\delta)}{\Gamma^2(\frac{3}{2}-\delta)} = \frac{\Sigma^{(1)}}{\pi}\frac{\Gamma^2(1-\delta)}{\Gamma^2(\frac{3}{2}-\delta)}. \tag{183}$$

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